QUANTIFICATION OF UNCERTAINTY USING BAYESIAN APPROACHES

Prof. Ezio Todini
President Italian Hydrological Society
University of Bologna - Italy

SCOPE

This presentation aims at clarifying the concept of

Predictive Uncertainty (PU)

and at providing a methodological framework for quantifying and using it in flood forecasting and water resources management.
Problems Involved

1. Flood Warning and Evacuation Management

2. Flood Detention and Diversion

3. Real Time Reservoir Management

Etc.

DECISION MAKING UNDER UNCERTAINTY

The Reservoir Management Problem

\[ E\{g(x)\} \neq g(E\{x\}) \]
**DECISION MAKING UNDER UNCERTAINTY**

The Linear Losses Case

\[
\text{Rel} = \mu(V) - V_{\text{max}} + \sigma(V) f(c)
\]

**THE DEFINITION OF PREDICTIVE UNCERTAINTY**

In order to understand the meaning of Predictive Uncertainty, let me pose the following question:

Flooding damages will occur:

(1) when the forecasted level overtops the dykes?

or

(2) when the actual future water level overtops the dykes?

The obvious answer is

(2) when the actual future water level overtops the dykes
THE DEFINITION OF PREDICTIVE UNCERTAINTY

This answer has a strong implication in the definition of PU.

PU is obviously the uncertainty that we have on the occurrence of a real future value, as for instance the water level in 12 hours from now.

This must not be confused with “Validation Uncertainty”.

THE DEFINITION OF PREDICTIVE UNCERTAINTY

Following Rougier (2007),

Predictive uncertainty

is the expression of a subjective assessment of the probability of occurrence a future (real) event conditional upon all the knowledge available up to the present (the prior knowledge) and the information that can be acquired through a learning inferential process.
VALIDATION UNCERTAINTY vs PREDICTIVE UNCERTAINTY

Meteorological Ensembles are a measure of the Validation Uncertainty, while Climatological distributions or Extreme Value distributions are measures of Predictive Uncertainty, although non conditional on real time information.

An example using Meteorological Ensembles
The definition of Validation Uncertainty

The **Validation Uncertainty** is the probability that the model **predicted value** (water level, discharge, water volume, etc) will be smaller or equal to a prescribed value

\[ \text{Prob}(\hat{y}_t \leq y^* | y_t, M, D_{\text{hist}}) \]

Please note that this expression cannot be used beyond time \( t \) because observations are not available.

The use of Validation Uncertainty

Assessment of Validation Uncertainty (VU) is essential to evaluate the performances of a model in order to improve it.

Therefore, when dealing with VU, one must also assess and separate the effects of **model uncertainty**, **parameter uncertainty**, input and output measurement uncertainty, initial and boundary conditions uncertainty.
Representation of Validation Uncertainty

$$\text{Prob}\{y_t = y^*_t\} = 1$$

The definition of Predictive Uncertainty

The **Predictive Uncertainty** is the probability that a future value of the **predictand** (water level, discharge, water volume, etc) will be smaller or equal to a prescribed value.

$$\text{Prob}(y_{t+k\Delta t} \leq y^*_t | \hat{y}_{t+k\Delta t}, M, \mathcal{D}_{\text{hist}})$$

given our prior knowledge, all the **historical information** and the **model forecast**
Real Time Flood Forecasting
and Warning Systems

Aims of Course

Representation of Predictive Uncertainty

\[ \text{Prob} \left\{ \hat{y}_{\phi_0} = \hat{y}^* \right\} = 1 \]

The use of Predictive Uncertainty

Assessment of Predictive Uncertainty is fundamental to take a decision given a model (or several models) forecast.

When using PU it is not necessary to assess and separate all the sources of errors if the conditional density used is consistent with the model(s) and all the other sources of uncertainty, which affected its development.
For a given model and a set of parameters one can derive predictand and model joint/conditional probability densities.

For a given model there are as many joint and conditional distributions as the number of parameter sets.

\[
\text{Prob}(y_{t+k,\Delta t} \leq y^* | \hat{y}_{t+k,\Delta t}(v^*_t), M, H_{\text{hist}})
\]
MODEL AND PARAMETER UNCERTAINTY

Therefore one must derive the “Posterior Density (PD)” of parameters \( g_\varphi (\varphi | \theta, \lambda_{\text{hist}}) \) using the classical Bayesian Inference. This PD is then used to marginalise, namely to integrate out, the effect of parameters.

In a continuous domain:

\[
F(y_{i+k}, \theta, \lambda_{\text{hist}}) = \int_{\varphi} F(y_{i+k}, \hat{y}_{i+k}, (\varphi), \theta, \lambda_{\text{hist}}) g_\varphi (\varphi | \theta, \lambda_{\text{hist}}) \, d\varphi
\]

or in discrete mode:

\[
F(y_{i+k}, \theta, \lambda_{\text{hist}}) = \sum_i F_i(y_{i+k}, \hat{y}_{i+k}, (\varphi_i), \theta, \lambda_{\text{hist}}) g_\varphi (\varphi_i | \theta, \lambda_{\text{hist}})
\]

MODEL AND PARAMETER UNCERTAINTY

Please note that this is TOTALLY different from what is proposed in Generalized Likelihood Uncertainty Estimation (GLUE), where the definition of PU is given as:

\[
P(\hat{Z}_t < z) = \sum_{i=1}^{B} L[M(\varphi_i) | \hat{Z}_{t,i} < z]
\]

where \( L = g_\zeta (\varphi_i | \theta, \lambda_{\text{hist}}) \) is nothing else than the posterior parameter density.

Where are the conditional predictive density (???)
as well as the marginalisation of parameter uncertainty (???)
Nonetheless, marginalising parameter uncertainty, although statistically correct, does not produce substantial differences from using a best fit parameter set. This is mostly due to the fact that the nearly best parameters produce predictions that are closely related among them, while the posterior probability of the worst parameter sets is obviously very low.
Aims of Course

Predictive uncertainty in hindcast mode
(Results on a Chinese catchment)

The difference between the marginalised density and the one obtained using the “best parameter set” can be relatively small

Predictive uncertainty in hindcast mode
(Results on a Chinese catchment)

Solid: Marginalised
Dashed: Max Likelihood
MODEL AND PARAMETER UNCERTAINTY

When the behaviour of a set of conditions such as errors deriving from the different sources varies at random in time in an “unpredictable manner” then one can use the “mixture of models” concept.

Please bear in mind that if the conditions ARE predictable then one is better off by using the “model” which best fits the observations under the relevant conditions.
This is why it is more interesting to approach the problem in terms of few alternative models of a widely different nature. 
i.e. a physically based model, a conceptual model and a data driven model.
This has given rise to the development of several multi-model Predictive Uncertainty Processors.

For different models there are as many joint and conditional distributions as the number of models.

\[ \text{Prob}\left( y_{t+k,d} \leq y^* | \hat{y}_{t+k,d} (M_i), \mathcal{D}_{\text{hist}} \right) \]
The Binary Response Processors convert continuous measurements and/or forecasts into discrete 0-1 probability of occurrence of one event.

- The Logit (based on the Logistic Distribution)
- The Probit (based on the Inverse Gaussian Distribution)
- The Bayesian Multivariate Binary Processor (BMBP)
- The Mixture of Beta Distributions

Useful tools, but the reliability of the continuous processors seems to be higher.
The Continuous Single or Multi-model Predictive Uncertainty Processors

Hydrological Uncertainty Processor
Krzysztofowicz, R., 1999; Krzysztofowicz and Kelly, 2000

Bayesian Model Averaging
Raftery et al., 2003;

Model Conditional Processor
Todini, 2008.


Krzysztofowicz Bayesian Processor
Krzysztofowicz (1999) approach (HUP) was the first to be developed in hydrological applications. It combines prior information embedded into an AR1 model with that deriving from a predictive model of unspecified nature (physically based, conceptual, etc.) Unfortunately

-It has a scalar formulation: only one model can be combined at a time
-The AR1 model is implicitly assumed to be independent from the predictive model
AVAILABLE PREDICTIVE UNCERTAINTY PROCESSORS
Raftery Bayesian Model Averaging

BMA aims at assessing the unconditional mean and variance of any future value of a predictand on the basis of several model forecasts.

\[
E\{y|M_i, \omega\} = \sum_{i=1}^{n} w_i E\{y|M_i\}
\]

\[
\text{Var}\{y|M_i, \omega\} = \sum_{i=1}^{n} w_i \text{Var}\{y|M_i\} + \sum_{i=1}^{n} w_i \left( \hat{y}_i - \sum_{i=1}^{n} w_i E\{y|M_i\} \right)^2
\]

It reformulates the Bayesian mixture equation

\[
F\{y|M_i, \omega, \omega_\text{hat}\} = \sum_{i=1}^{n} F\{y|M_i\} \text{Prob}\{M_i|\omega, \omega_\text{hat}\}
\]

by considering the posterior probability as a weight

\[
\text{Prob}\{M_i|\omega, \omega_\text{hat}\} = w_i.
\]

AVAILABLE PREDICTIVE UNCERTAINTY PROCESSORS

The BMA weights are estimated by constrained maximisation of the Likelihood of observing the predictand

\[
\begin{align*}
\max_{w_i} \log L & = \sum_{i=1}^{n} \log \left( \sum_{i=1}^{n} w_i p_i(y_i | \hat{y}_i) \right) \\
\text{s.t.} & \quad \sum_{i=1}^{n} w_i = 1; \quad w_i \geq 0 \quad \forall i = 1, \ldots, n
\end{align*}
\]

on the assumption that the probability densities of the observations as well as of the model forecasts are all approximately Gaussian, which is correct if using the Normal Quantile Transform (NQT) to transform the data.
AVAILABLE PREDICTIVE UNCERTAINTY PROCESSORS

The Model Conditional Processor

If one makes the hypothesis that all the NQT transformed variables follow a multi-Gaussian joint probability density, a more natural approach can be:

- To develop a set of models in the real untransformed space (one or more than one)
- To build the joint probability density in the Gaussian space (Predictand, a priori model, deterministic model, etc.)
- To simply compute the probability of the predictand conditional on ALL the model predictions

A Useful Property of the Multivariate Normal Distribution

Given a vector \( y \) of Normally distributed random variables, with Mean \( \mu_y \) and Variance \( \Sigma_y \), with marginal distributions \( y \sim N(\mu_y, \Sigma_y) \) and \( \hat{y} \sim N(\hat{\mu}_y, \hat{\Sigma}_y) \), then the conditional distribution is also a Normal distribution

\[
N(\mu_{y|\hat{y}}, \Sigma_{y|\hat{y}})
\]

with conditional Mean

\[
\mu_{y|\hat{y}} = \mu_y + \Sigma_{y\hat{y}} \Sigma_{\hat{y}\hat{y}}^{-1} (\hat{y} - \hat{\mu}_y)
\]

and conditional Variance

\[
\Sigma_{y|\hat{y}} = \Sigma_{y\hat{y}} - \Sigma_{y\hat{y}} \Sigma_{\hat{y}\hat{y}}^{-1} \Sigma_{\hat{y}y}
\]
AVAILABLE PREDICTIVE UNCERTAINTY PROCESSORS

Advantages of MCP

- It allows one to combine together a wide variety of different models without the need of using the constrained optimisation required by BMA

- It accounts for correlation among the predictive models

- It allows one to have multiple outputs, benefitting from spatial correlation (for instance several water levels along the same river)

MODEL CONDITIONAL PROCESSOR

THE NEED FOR USING THE TRUNCATED NORMAL DISTRIBUTION

In many cases the statistical pattern and the correlation between observed and modelled quantities differs from lower to higher values. In this case the estimated predictive uncertainty is affected by the non stationarity of the errors.

Correlation may be quite different if one considers high or low water stages. Particularly when using flood routing models, lower water level forecasts are highly influenced by the lack of proper knowledge of the river geometrical description.
In order to overcome this problem MCP can also be defined in terms of the Bivariate Truncated Normal Distribution. Higher and lower values are then treated using two different Binary Truncated Normal Distributions.

**COMPARISON OF DIFFERENT APPROACHES**

The river Po in Italy

*Results obtained in collaboration with ARPA-SIM of Emilia-Romagna*

**Pontelagoscuro**

Basin size = 70,000 km²
EXAMPLES
Hydraulic model + AR1

Bias

Variance

EXAMPLES
Hydraulic model + AR1 + NN

Variance

% observations within limits
Po at Pontelagoscuro: 36 h forecast
Mike11 and MCP processed PU

MISSED ALARMS CORRECTED BY THE MCP

FALSE ALARM CORRECTED BY THE MCP
Po at Pontelagoscuro: 36 h forecast
Mike11 + ANN and MCP processed PU

STATISTICAL INDEXES FOR THE VALIDATION PERIOD (01/01/2004 – 30/01/2009)

CONTINUOUS FORECAST STATISTICS

<table>
<thead>
<tr>
<th></th>
<th>ERROR MEAN</th>
<th>ERROR VARIANCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>MODEL</td>
<td>0.1803</td>
<td>7.23E-02</td>
</tr>
<tr>
<td>HM + MCP</td>
<td>0.0157</td>
<td>5.09E-02</td>
</tr>
<tr>
<td>HM + ANN + MCP</td>
<td>-6.56E-03</td>
<td>3.98E-02</td>
</tr>
</tbody>
</table>

EVENT STATISTICS

<table>
<thead>
<tr>
<th></th>
<th>NEGATIVE HITS</th>
<th>FALSE ALARMS</th>
<th>MISSED ALARMS</th>
<th>POSITIVE HITS</th>
</tr>
</thead>
<tbody>
<tr>
<td>MODEL</td>
<td>42</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>HM + MCP</td>
<td>43</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>HM + ANN + MCP</td>
<td>44</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

RESULTS

BORETTO (PO) – Forecasting horizon = 36 h

FIUME PO - BORETTO - ORIZZONTE DI PREVISIONE 36 h
LIVELLI PREVISTI E PROBABILITA' DI SUPERAMENTO DI UNA SOGLIA
RESULTS
BORETTO (PO)

Boreto - Calibrazione

Boreto - Validazione

MCP: APPLICATION
BARON FORK RIVER AT ELDON, OK, USA

TOPKAPI MODEL

TETIS MODEL

Available data, provided by the NOAA’s National Weather Service, within the DMIP 2 Project:

Gridded hourly precipitation and temperature data
Observed hourly discharge at Eldon

© Newcastle University 2010
Bayesian Combination of Different Hydrological Models

STD

Time delay (Hours)

Q (m³/s)

TPK
TPK+MCP
TET
TET+MCP
ANN
ANN+MCP
TPK+TET+MCP
TPK+ANN+MCP
TET+ANN+MCP
TPK+TET+ANN+MCP

NS

Time delay (Hours)

© Newcastle University 2010
CONCLUSIONS (1/3)

Time has passed since the concept of Predictive Uncertainty was introduced in Bayesian Statistics.

Unfortunately, limited operational use of PU can be found in the fields of Flood Forecasting, Flood Emergency Management and Water Resources Management mostly due to the widespread confusion on the PU definition and concepts.

CONCLUSIONS (2/3)

Whereas the use of HUPs for operational purposes is in progress, the use of Meteorological QPF for the estimation of PU has not yet reached a reasonable level of acceptance. This is due to two main reasons:
1) The first one is due to the lack of understanding of the operational use and of the real benefits deriving from incorporating PU in the decision process.
2) The second one relates to the “lack of will” shown by the meteorological organisations when requested to re-run their models on past data.
CONCLUSIONS (3/3)

In practice we hydrologists have the following homework: we must convince stakeholders and meteorologists of the real benefits deriving from the estimation and use of Predictive Uncertainty in flood warning, flood emergency and water resources management.

Thank you for your patience and attention.